Lévy models in finance: Lecture II

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Summary

General aim: describe jump modeling in finace through some relevant issues.

- Lecture 1: Black-Scholes model
- Lecture 2: Models with jumps
- Lecture 3: Optimal stoping for processes with jumps
- Lecture 4: Symmetry and skewness in Lévy markets

Alternatives to Black Scholes (BS)

We begin with two remarks:

- Asset prices do not verify the statistical assumptions of Black and Scholes model, in particular normal distribution, skewness, kurtosis, independence.
- Prices of options observed in the market do not give a consistent value for the volatility in Black and Scholes formula. This fact is known as the “smile phenomena”.

In conclussion, several atempts have been done in order to obtain models that incorporate this empirical facts. We distinguish two directions for the generalization of BS model:

Stochastic volatility models

- they loose the independence of increments for log-stock prices of BS,
- but they preserve the continuity of paths, that follow an equation of the type

$$dS_t = S_t\big(\mu dt + \sigma_t dW_t\big)$$

For σ_t we have two possibilities

- $\sigma_t = \sigma(S_t, t)$ called *local volatility models*
- σ_t^2 follows a SDE itself:

$$d\sigma_t^2 = \alpha(\sigma_t^2)dt + \beta(\sigma_t^2)d\bar{W}_t$$

where $\bar{W}$ is another brownian motion (possibly) correlated with W. This models are popular, and called *stochastic volatility models*.

Models with jumps

- They loose the continuity of trajectories, incorporating jumps,
- but they preserve the independence and stationarity of the increments of the log-prices, through the model

$$S_t = S_0 e^{X_t}$$

where X is a Lévy process, i.e. a stochastic process with stationary and independent increments.

Definition: Lévy processes

$X = (X_t)_{t \geq 0}$ is a Lévy process defined on $(\Omega, \mathcal{F}, P)$ when

- $X_0 = 0$, i.e. it starts at the origin.
- X has trajectories with limits from the left and are continuous from the right ("cadlag")
- Its increments are independent: if $0 \leq t_1 \leq \cdots \leq t_n$, then

$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

  are independent random variables.
- It increments are stationary:

$$X_t - X_s \sim X_{t-s}.$$

Lévy-Kinchine formula

In order to study Lévy process we have an analytical tool of great relevance, given by the Lévy-Kinchine formula, that states:

$$E(e^{zX_t}) = e^{t\psi(z)},$$

where the cumulant or stochastic exponent ψ is given by the formula

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}})\Pi(dy).$$

Here

- b and $\sigma \geq 0$ are real numbers,
- Π is called the *jump measure*, being a postive measure defined on $\mathbf{R} - \{0\}$, such that $\int(1 \wedge y^2)\Pi(dy) < +\infty$.

Triplet of LP process

We then have a triplet

$$(b, \sigma, \Pi).$$

that characterizes the law of the Lévy process (This fact is the Lévy-Khinchine Theorem).

We conclude that all the probabilistic information of the process is contained in the triplet, and (it can also be shown) also is characterized by the cumulant of the process.

Ths makes possible to study *probabilistic* properties of LP with *analytic* mathematical tools, as functions of complex variable

Some analytic computations:

Let us compute the moments of a LP.

$$E(X_t) = t\psi'(0). \tag{1}$$

In fact, derivating $E(e^{zX_t}) = e^{t\psi(z)}$ with respect to z, we obtain

$$E(X_t e^{zX_t}) = e^{t\psi(z)}\psi'(z),$$

that evaluated at $z = 0$ gives the formula (1).

Similarly, we obtain

$$E(X_t^2) = t\psi''(0).$$

We now write this values in term of the triplet of the process: Derivating

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}})\Pi(dy)$$

we arrive to

$$\psi'(z) = b + \sigma^2 z + \int_{\mathbf{R}} (ye^{zy} - y\mathbf{1}_{\{|y|<1\}})\Pi(dy),$$

from where we deduce that

$$E(X_1) = \psi'(0) = b + \int_{\mathbf{R}} y\mathbf{1}_{\{|y|\geq 1\}}\Pi(dy).$$

Similarly we obtain

$$E(X_1^2) = \sigma^2 + \int_{\mathbf{R}} y^2\Pi(dy).$$

Lévy markets

We have two assets:

- A savings account
$$B_t = B_0 e^{rt}$$
as in BS, and
- A risky asset, of the form
$$S_t = S_0 e^{X_t}$$
where $X = (X_t)_{t \geq 0}$ is a Lévy process.

Example: Black Scholes model

The log-price process X is

$$X_t = \sigma W_t + (\mu - \frac{1}{2}\sigma^2)t,$$

that is a Lévy process with triplet

$$(\mu - \frac{1}{2}\sigma^2, \sigma, 0).$$

The absence of jumps can be read in the fact that

$$\Pi = 0.$$

Example: Poisson process

Let $T_1, T_2, \dots$ be independent random variables with exponential distribuion and parameter λ. Let

$$N_t = \inf\{k : T_1 + T_2 + \dots T_k \leq t\}.$$

$N = (N_t)_{t \geq 0}$ is a Poisson process with parameter λ.

$$X_t = bt + cN_t$$

is a Lévy process with triplet $(b, 0, \Pi)$, where

$$\Pi(dy) = \lambda \delta_c(dy).$$

$\sigma = 0$ corresponds to the fact that there is no brownian component.

Example: Compound Poisson Process

Let us consider $T_1, T_2, \dots$ as before, and $Y = \{Y_k\}_{k \in \mathbf{N}}$ a sequence of independent random variables with common distribution $F = F(y)$. We construct

$$X_t = bt + \sum_{k=1}^{N_t} Y_k,$$

The triplet of this Lévy process is

$$(b, 0, \lambda F(dy)).$$

If $Y = c$ (constant), we have $F(dy) = \delta_c(dy)$ and that is the previous example (the Poisson process).

Example: Diffusion with jumps

Consider as before

- N a Poisson process,
- $Y = \{Y_n\}_{n\in\mathbf{N}}$ a sequence of independent random variables with common distribution $F(y)$
- W, a brownian motion,

to construct $X = (X_t)_{t\geq 0}$ given by

$$X_t = bt + \sigma W_t + \sum_{k=1}^{N_t} Y_k.$$

X has triplet

$$(b, \sigma, \lambda F(dy))$$

Example: Merton Jump Diffusion model

In 1976 Merton introduced the first diffusion with jumps model, assuming that the jump distribution is a gaussian random variable. We then have that Y_k are gaussian. We then have

$$F(dy) = \frac{1}{\delta\sqrt{2\pi}} e^{-(x-\nu)^2/(2\delta^2)} dy.$$

The jump measure in this case is $\lambda F(dy)$, and the cumulant is

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \lambda\left(e^{\nu z + \delta^2 z^2/2} - 1\right)$$

If $\lambda = 0$ we do not have jumps, and obtain the classsical BS model.

Example: Double exponential (Kou) model

Kou model assumes that the jumps have an asymetric double exponential distribution. More precisely, the common density of the random varialbes Y_k is

$$f(dy) = \begin{cases} p\alpha e^{-\alpha y}, & \text{si } y > 0, \\ (1-p)\beta e^{\beta y}, & \text{si } y < 0. \end{cases}$$

The characteristic exponent in this case is

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \lambda \left(\frac{pz}{\alpha - z} - \frac{(1-p)z}{\beta + z} \right).$$

Pricin options in jump models

We have the following equivalences:

- *The market is complete*
- *There exists a perfect hedging*
- *The risk-netrual measure is unique*
- *There exists a rational price*

But model with jumps are called *incomplete* and are characterized by the following facts:

- *There is no perfect hedging*
- *There exists an infinite number of risk-neutral measures*
- *There exists infinite possible prices, named admissible prices.*

A probability measure Q is risk-neutral for the market model if

1. Q is equivalent to P, the historical or physical measure,
2. $S_t/B_t = S_0 e^{X_t - rt}$ is a Q-martingale.

In order to choose the risk-neutral pricing measure, there are several different possibilities, we mention some of them:

- In incomplete markets with continuous trajectories (as stochastic volatility models) Föllmer and Schweizer introduced the *minimal measure*, that minimizes the quadratic loss of a hedge: If π is a self-financing portfolio with capital V^π, corresponding to a payoff $f(S_T)$ this measure minimizes

$$\min_\pi E\left(\left(V_T^\pi - f(S_T)\right)^2\right)$$

- In the framework of Lévy processes, Gerber and Shiu proposed to consider the Esscher transform (coming from Actuarial mathematics), that minimizes the relative entropy (Chan, 1999).
- A Lévy process is stable under the Esscher transform: if X is a LP under P, then it is LP under Q, with characteristic exponent ψ_Q.

One pricing measure: Esscher Transform

Due to the convexity property of the cumulant, there exists θ such that

$$\psi_P(\theta + 1) - \psi_P(\theta) = r$$

The risk-neutral measure Q given by Esscher transform satisfies

$$\frac{dQ}{dP} = \exp(\theta X_T - \psi_P(\theta)T).$$

The process X under Q has exponent

$$\psi_Q(z) = \psi_P(z + \theta) - \psi_P(\theta).$$

and the martingale condition is

$$\psi_Q(1) = r.$$

Option Pricing

It is easy to compute an expectation when you know the density, that is not our case.
Here we present Lewis formula (2001), based on Parseval equality of Fourier transofrms. (See also Carr and Madan, 1999)

The option we price has the following characteristics:

- European type, excercise time T.
- Payoff function $f(S_T)$
- A call option has $f(x) = (x - K)^+$,
- A put option has $f(x) = (K - x)^+$.
- We assume that the pricing measure is Q given by the Esscher transform (for instance, but this assumption is not necessary to our computations)

Lewis Formula

The price of an European option with payoff $f(S_T)$ is given by

$$V(S_0, T) = \frac{e^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} \frac{1}{S_0^{iz}} E_Q(e^{-izX_T}) \hat{f}(z) dz.$$

Here

- The domain of integration is the line $\{z = i\nu + t, t \in \mathbb{R}\}$, in the complex plane, where $\nu > 1$ is such that the integrals that appear in our computations converge.
- $\hat{f}$ is the Fourier transform of the payoff function f:

$$\hat{f}(z) = \int_{i\nu-\infty}^{i\nu+\infty} e^{izx} f(x) dx.$$

If $f(x) = (x-K)^+$, then $\hat{f}(z) = -K^{1+iz}/(z^2 - iz)$.

- The formula is obtained applying the Parseval identity (that holds in Hilbert spaces):

$$\int_{-\infty}^{\infty} f(x)p(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(u)\hat{p^*}(u)du,$$

(${z^*}$ is the conjugate, valid under certain conditions). We apply this formula to the density $p(u)$ of S_T

$$\hat{p^*}(u) = \int_{-\infty}^{\infty} e^{-izu}p(u)du = (S_0)^{-iz}E_Q e^{-izX_T},$$

- The numerical computation is performed using the Fast Fourier transform(FFT) after the following transformations
- Similar arguments give the Carr and Madan (1999) formula.

Application of Lewis formula

Let us consider an european call in the Merton model. We have

- $E_Q(e^{-iZX_T}) = e^{T\psi(-iz)} = \exp\left(-ibz - \frac{1}{2}\sigma^2 z^2 + \lambda\left(e^{-i\nu z - \delta^2 z^2/2} - 1\right)\right)$
- $f(x) = (x-K)^+$ por lo que $\hat{f}(z) = -K^{1+iz}/(z^2 - iz)$

We have

$$V(S_0, T) = \frac{-Ke^{-rT}}{2\pi}\int_{i\nu-\infty}^{i\nu+\infty} \frac{(K/S_0)^{iz}}{z^2 - iz} \exp\left(-ibz - \frac{1}{2}\sigma^2 z^2 + \lambda\left(e^{-i\nu z - \delta^2 z^2/2} - 1\right)\right) dz$$

Particular case: Black Scholes model

If $\lambda = 0$ in Merton model we recover BS model. In this case $b = r$ due to the martingale condition. The option price is

$$\begin{aligned} V(S_0, T) &= \frac{-Ke^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} \frac{(S_0/K)^{-iz}}{z^2 - iz} e^{-irz-\frac{1}{2}\sigma^2 z^2} dz \\ &= \frac{-Ke^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} \frac{e^{-ikz}}{z(z-i)} e^{-irz-\frac{1}{2}\sigma^2 z^2} dz \\ &= \frac{-Ke^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-ikz} \left(\frac{i}{z} - \frac{i}{z-i} \right) e^{-\frac{1}{2}\sigma^2 z^2} dz \end{aligned}$$

where $k = \log(S_0/K) + rT$. Each term gives the corresponding term in BS formula based on the calculus of residuales (the detalis can be found in Lewis (2000)).

FFT in Lewis Formula

We compute, denoting $k = \log S_0 + rT$ and $X_t^r = X_t - rt$, the values of

$$V(S_0, T, k) = \frac{e^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-ikz} E_Q(e^{-izX_T^r}) \hat{f}(z) dz,$$

that are of the form

$$V(k) = \int_{i\nu-\infty}^{i\nu+\infty} e^{-ikz} g(z) dz$$

FFT gives the Fourier transform of a discrete sequence, and by this reason we make the following approximations:

$$\int_{i\nu-\infty}^{i\nu+\infty} e^{-ikz} g(z)dz \approx \int_{i\nu-A/2}^{i\nu+A/2} e^{-ikz} g(z)dz$$

$$\approx \frac{A}{N} \sum_{j=0}^{N-1} w_j g(z_j) e^{-ikz_j}$$

where

- $z_j = -A/2 + j\Delta \quad (j = 0, \ldots, N-1)$
- $\Delta = A/(N-1)$
- w_j are the weights corresponding to an integration rule, for instance in the classical parallelogram rule, the weights are

$$w_0 = w_{N-1} = 1/2, \quad w_j = 1 \ (j = 1, \ldots, N-2).$$

Now, putting $k = k_n = 2\pi n/(N\Delta) = \log S_{0,n} + rT$ (a vector of initial stock prices), the sum is transformed in order to apply the FFT:

$$\frac{A}{N}\sum_{j=0}^{N-1} w_j g(z_j) e^{-i(2\pi n/(N\Delta))(-A/2+j\Delta)}$$

$$= \frac{A}{N} e^{iAu_n/2} \sum_{j=0}^{N-1} w_j g(z_j) e^{-i(2\pi jn/N)}$$

In conclussion, the FFT allows us to compute efficiently a vector of prices for a vector of initial prices of the form

$$S_{0,n} = \exp\left(\frac{2\pi n}{N\Delta} - rT\right)$$

Other parametrizations, allow to compute prices for a vector of strikes.

American options

An american options is a contract that pays a payoff $f(S_\tau)$ at time $\tau \in [0, T]$ that can be choosen by the holder of the option.

When the option has no expiration time, i.e. $(T = \infty)$ it is named a perpetual option.

The price of an american option with finite maturity $(T < \infty)$ is obtained solving an optimal stopping problem, that usualy does not admit closed form solution, and is carried out with the help of numerical methods.

In the perpetucal case, there exists some closed formuals. (Mc Kean (1965), Merton (1973) for BS model)
We now present some results for LP processes.
Let us choose Q such that the process X:generalize this two result.

- is a Lévy process under Q,
- $\Big(S_t/B_t\Big)_{t\geq 0}$ is a martingale under Q

In order to price the perpetual put option the following optimal stopping problem must be solved:

$$P(S_0) = \sup_{\tau \in \mathcal{M}} E_Q\big(e^{-r\tau}(K - S_\tau)^+\big),$$

and similarly for the call $C(S_0)$. The result is the following

Theorem.(M. 2002) $S_t = S_0 e^{X_t}$, X is a LP. Set

$$I = \inf\{X_t \colon 0 \le e_r\}, \quad M = \sup\{X_t \colon 0 \le e_r\}.$$

where e_r is $exp(r)$ independent of X. The price of the perpetual put option and call option with dividens for the stock model is given by

$$P(S_0) = \frac{E(K - S_0 e^I)^+}{E(e^I)}, \quad C(S_0) = \frac{E(S_0 e^M - K)^+}{E(e^M)}$$

and the optimal excercise time is given by

$$\begin{aligned}\tau_P^* &= \inf\{t \ge 0 \colon S_t \le S_0 E(e^I)\},\\ \tau_C^* &= \inf\{t \ge 0 \colon S_t \ge S_0 E(e^M)\}.\end{aligned}$$

New problem: Compute the distribution of I and M.

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